STUDY MATERIALS

(VECTOR CALCULUS)

DIFFERENTIAL GEOMETRY

Mathematics Honours
Semester – 2
Paper – C4T Unit - 4

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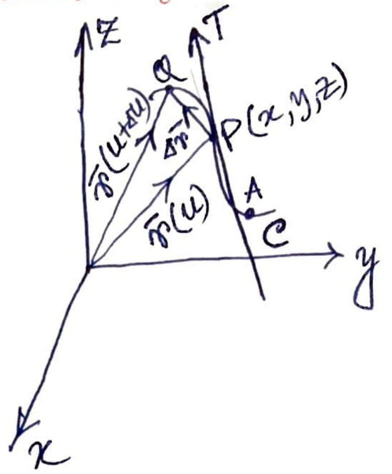
Tangent Vector to a space curve :-



Let $\bar{r}(u)$ be the position vector of any point $P(x,y,z)$ to the space curve C having parametric equations:

$x = x(u)$, $y = y(u)$, $z = z(u)$.

$\therefore\ \bar{r}(u) = x(u)\hat{i} + y(u)\hat{j} + z(u)\hat{k}$

Let $Q(x+\Delta x, y+\Delta y, z+\Delta y)$ be any point neighbouring P on C. If the position vector of Q be $\bar{r}(u+\Delta u) = \bar{r} + \Delta\bar{r}$, then $\overline{PQ} = \overline{OQ} - \overline{OP}$

$\Rightarrow \overline{PQ} = \bar{r}(u+\Delta u) - \bar{r}(u) = \bar{r} + \Delta\bar{r} - \bar{r} = \Delta\bar{r}$.

Then $\lim\limits_{\Delta u\to 0} \dfrac{\overline{PQ}}{\Delta u} = \lim\limits_{\Delta u\to 0} \dfrac{\Delta\bar{r}}{\Delta u} = \lim\limits_{\Delta u\to 0} \dfrac{\bar{r}(u+\Delta u) - \bar{r}(u)}{\Delta u}$

$= \dfrac{d\bar{r}}{du}$; if the limit exists.

$\dfrac{d\bar{r}}{du}$ is the vector in the direction of the tangent to the space curve C at (x,y,z) and is given by

$\dfrac{d\bar{r}}{du} = \dfrac{dx}{du}\hat{i} + \dfrac{dy}{du}\hat{j} + \dfrac{dz}{du}\hat{k}$; $\dfrac{d\bar{r}}{du}$ is called a <u>tangent vector</u> to C at (x,y,z).

Now, if the curve C is defined by parametric equations $x = x(s)$, $y = y(s)$, $z = z(s)$, where s is the arc length of $\overset{\frown}{AP}$ measured from a fixed point A on C.

Then $\bar{r}(s) = x(s)\hat{i} + y(s)\hat{j} + z(s)\hat{k}$.

$\therefore\ \dfrac{d\bar{r}}{ds} = \dfrac{dx}{ds}\hat{i} + \dfrac{dy}{ds}\hat{j} + \dfrac{dz}{ds}\hat{k}$; $\left|\dfrac{d\bar{r}}{ds}\right| = \sqrt{\left(\dfrac{dx}{ds}\right)^2 + \left(\dfrac{dy}{ds}\right)^2 + \left(\dfrac{dz}{ds}\right)^2}$

$\Rightarrow \left|\dfrac{d\bar{r}}{ds}\right| = \sqrt{\dfrac{(dx)^2 + (dy)^2 + (dz)^2}{(ds)^2}} = 1$ $\left[\because (dx)^2 + (dy)^2 + (dz)^2 = (ds)^2 \text{ (from calculus)}\right]$.

$\therefore\ \dfrac{d\bar{r}}{ds}$ is a unit tangent vector to C.

From the above, we have the tangent vector to C at (x,y,z) is given by $\dfrac{d\bar{r}}{dt}$, if $x = x(t)$, $y = y(t)$, $z = z(t)$.

$\therefore\ \dfrac{\frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|} = \dfrac{d\bar{r}}{ds} \Rightarrow \dfrac{d\bar{r}}{dt} = \dfrac{d\bar{r}}{ds}\cdot\left|\dfrac{d\bar{r}}{dt}\right| \longrightarrow (1)$

Also, $\dfrac{d\bar{r}}{dt} = \dfrac{d\bar{r}}{ds}\cdot\dfrac{ds}{dt} \longrightarrow (2)$; Comparing (1) & (2), we get $\dfrac{ds}{dt} = \left|\dfrac{d\bar{r}}{dt}\right|$.

Ex. Find the unit tangent $\bar{T}$ to any point on the space curve: $x = t - t^3/3$, $y = t^2$, $z = t + t^3/3$.

A tangent vector to the curve at any point is

$$\frac{d\bar{r}}{dt} = \frac{d}{dt}\left[\left(t - \frac{t^3}{3}\right)\hat{i} + t^2\hat{j} + \left(t + \frac{t^3}{3}\right)\hat{k}\right] = (1-t^2)\hat{i} + 2t\hat{j} + (1+t^2)\hat{k}$$

$$\left|\frac{d\bar{r}}{dt}\right| = \sqrt{(1-t^2)^2 + (2t)^2 + (1+t^2)^2} = \sqrt{2(1+t^4) + 4t^2} = \sqrt{2}\,(1+t^2)$$

$\therefore$ Unit tangent $\bar{T} = \dfrac{d\bar{r}}{dt} \Big/ \left|\dfrac{d\bar{r}}{dt}\right| = \dfrac{(1-t^2)\hat{i} + 2t\hat{j} + (1+t^2)\hat{k}}{\sqrt{2}\,(1+t^2)}$

Note: Since $\left|\dfrac{d\bar{r}}{dt}\right| = \dfrac{ds}{dt}$, $\bar{T} = \dfrac{d\bar{r}}{dt} \Big/ \dfrac{ds}{dt} = \dfrac{d\bar{r}}{ds}$.

Ex. A particle moves along the curve $\bar{r} = (t^3 - 4t)\hat{i} + (t^2 + 4t)\hat{j} + (8t^2 - 3t^3)\hat{k}$, t is the time. Find the magnitude of the tangential component of its acceleration when $t = 2$.

Velocity, $\bar{v} = \dfrac{d\bar{r}}{dt} = (3t^2 - 4)\hat{i} + (2t + 4)\hat{j} + (16t - 9t^2)\hat{k}$

Accl^n., $\bar{a} = \dfrac{d\bar{v}}{dt} = \dfrac{d^2\bar{r}}{dt^2} = 6t\hat{i} + 2\hat{j} + (16 - 18t)\hat{k}$

$$\left[\frac{d\bar{r}}{dt}\right]_{at\ t=2} = 8\hat{i} + 8\hat{j} - 4\hat{k} = 4(2\hat{i} + 2\hat{j} - \hat{k}) = \bar{v}$$

$$\left[\frac{d^2\bar{r}}{dt^2}\right]_{at\ t=2} = 12\hat{i} + 2\hat{j} - 20\hat{k} = \bar{a}$$

$$v = \frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right| = \sqrt{4^2(2^2 + 2^2 + 1^2)} = 4 \cdot 3 = 12.$$

$$\bar{T} = \frac{d\bar{r}}{ds} = \frac{d\bar{r}}{dt} \Big/ \frac{ds}{dt} = \frac{4}{12}(2\hat{i} + 2\hat{j} - \hat{k}) = \frac{1}{3}(2\hat{i} + 2\hat{j} - \hat{k}).$$

Tangential component of $\bar{a} = \bar{a} \cdot \bar{T}$

$$= (12\hat{i} + 2\hat{j} - 20\hat{k}) \cdot \frac{1}{3}(2\hat{i} + 2\hat{j} - \hat{k})$$

$$= \frac{1}{3}(24 + 4 + 20) = \underline{\underline{16}}$$

DIFFERENTIAL GEOMETRY

Differential geometry involves a study of space curves and surfaces.

If the vector equation of the space curve C is given by $\bar{r} = \bar{r}(s)$, s is the arc length measured from some fixed point on C, then

$\frac{d\bar{r}}{ds} (\equiv \hat{t})$ is a unit tangent vector to C.

Now $\hat{t} \cdot \hat{t} = 1 \Rightarrow \frac{d}{ds}(\hat{t} \cdot \hat{t}) = 0 \Rightarrow 2\hat{t} \cdot \frac{d\hat{t}}{ds} = 0$

$\Rightarrow \hat{t} \cdot \frac{d\hat{t}}{ds} = 0 \Rightarrow \frac{d\hat{t}}{ds} \perp \hat{t}$, if $\frac{d\hat{t}}{ds} \neq \bar{0}$

If $\hat{n}$ is unit vector in the direction of $\frac{d\hat{t}}{ds}$ which is in normal direction, then

$\frac{d\hat{t}}{ds} = \kappa \hat{n}$, where κ is called the curvature of C at the specified point, and $\rho = 1/\kappa$ is called the radius of curvature there.

$\hat{n}$ is called the principal normal to C.

$\frac{d\hat{t}}{ds}$ is the measure of curvature of C.

Let $\hat{b}$ be a unit vector perpendicular to the plane of $\hat{t}$ and $\hat{n}$, and such that $\hat{b} = \hat{t} \times \hat{n}$.

Then $\hat{b}$ is called the binormal to C.

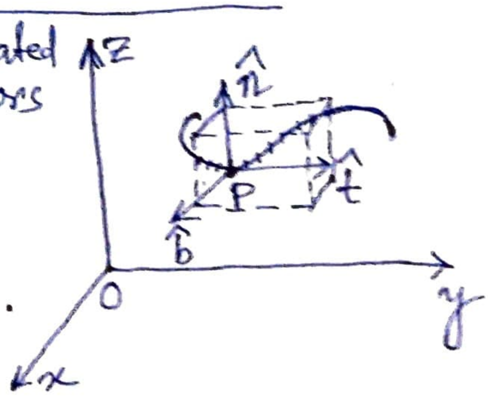
The directions $\hat{t}, \hat{n}, \hat{b}$ form a localized right-handed rectangular co-ordinate system at any specified point of C.

As s changes, this coordinate system moves and is known as the moving trihedral.

Three fundamental planes associated with three fundamental unit vectors $\hat{t}, \hat{n}, \hat{b}$ are: (i) Osculating plane,

(ii) Normal plane,

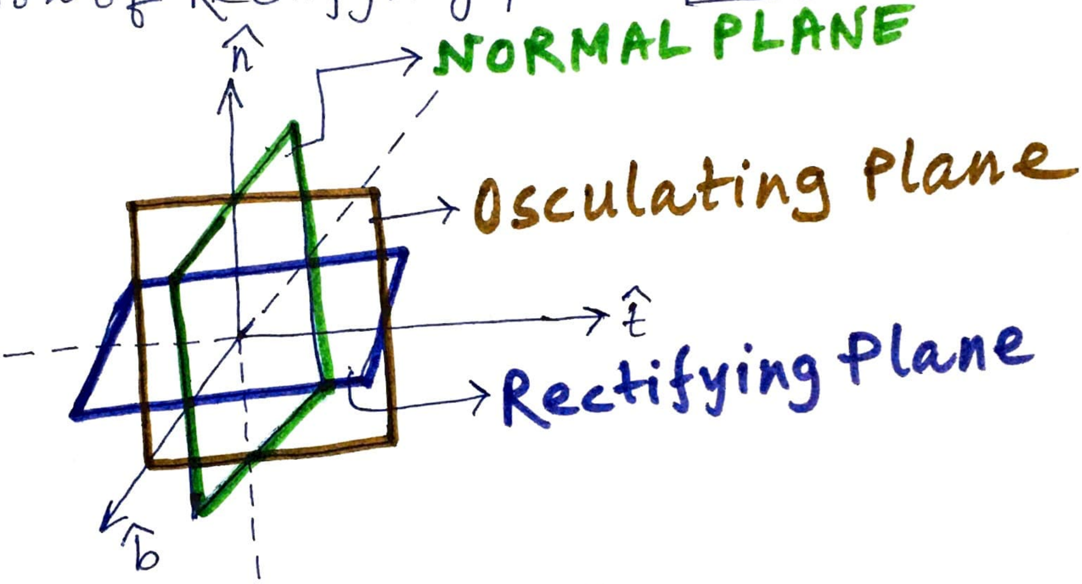
(iii) Rectifying plane.



(i) Osculating Plane to a curve C at a point P is the plane containing $\hat{t}$ and $\hat{n}$ at P.
$\therefore$ Equation of Osculating plane : $\boxed{(\bar{r}-\bar{r}_0)\cdot\hat{b}=0,}$
where $\bar{r}_0$ is the position vector of P.

(ii) Normal Plane to a curve C at a point P is the plane containing $\hat{n}$ and $\hat{b}$ at P.
$\therefore$ Equation of Normal Plane : $\boxed{(\bar{r}-\bar{r}_0)\cdot\hat{t}=0.}$

(iii) Rectifying Plane to a curve at a point P is the plane containing $\hat{t}$ and $\hat{b}$ at P.
$\therefore$ Equation of Rectifying plane : $\boxed{(\bar{r}-\bar{r}_0)\cdot\hat{n}=0}$



FRENET-SERRET Formulae :-

(i) $\frac{d\hat{t}}{ds}=\kappa\hat{n}$; (ii) $\frac{d\hat{n}}{ds}=\tau\hat{b}-\kappa\hat{t}$; (iii) $\frac{d\hat{b}}{ds}=-\tau\hat{n}$.

κ is the curvature of the space curve ;
τ is the torsion " " " "
$\rho=1/\kappa$ is called the radius of curvature.
$\sigma=\frac{1}{\tau}$ " " " " " torsion.

PROOF OF FRENET-SERRET FORMULAE:-

(i) $\frac{d\hat{t}}{ds} = \kappa\hat{n}$; (ii) $\frac{d\hat{n}}{ds} = \tau\hat{b} - \kappa\hat{t}$; (iii) $\frac{d\hat{b}}{ds} = -\tau\hat{n}$.

(i) $\hat{t}\cdot\hat{t} = 1$, so that $\frac{d}{ds}(\hat{t}\cdot\hat{t}) = 0 \Rightarrow 2\hat{t}\cdot\frac{d\hat{t}}{ds} = 0$

$\Rightarrow \hat{t}\cdot\frac{d\hat{t}}{ds} = 0 \Rightarrow \frac{d\hat{t}}{ds} \perp \hat{t} \Rightarrow \frac{d\hat{t}}{ds}$ is in normal direction.

If $\hat{n}$ is a unit vector in the normal direction, then $\frac{d\hat{t}}{ds} \parallel \hat{n} \Rightarrow \frac{d\hat{t}}{ds} = \kappa\hat{n}$, where κ is a scalar called the curvature of the curve at a given point. $\hat{n}$ is called the principal normal.

$\therefore \boxed{\frac{d\hat{t}}{ds} = \kappa\hat{n}}$ (is proved).

(iii) Let $\hat{b} = \hat{t}\times\hat{n}$, so that $\frac{d\hat{b}}{ds} = \hat{t}\times\frac{d\hat{n}}{ds} + \frac{d\hat{t}}{ds}\times\hat{n}$

or, $\frac{d\hat{b}}{ds} = \hat{t}\times\frac{d\hat{n}}{ds} + \kappa\hat{n}\times\hat{n}$ [using (i)]

$$= \hat{t}\times\frac{d\hat{n}}{ds} + \kappa\bar{0} = \hat{t}\times\frac{d\hat{n}}{ds}$$

$\Rightarrow \hat{t}\cdot\frac{d\hat{b}}{ds} = \hat{t}\cdot\hat{t}\times\frac{d\hat{n}}{ds} = 0 \Rightarrow \frac{d\hat{b}}{ds} \perp \hat{t}$.

Again, $\hat{b}\cdot\hat{b} = 1 \Rightarrow \frac{d}{ds}(\hat{b}\cdot\hat{b}) = 0 \Rightarrow \hat{b}\cdot\frac{d\hat{b}}{ds} = 0$

$\Rightarrow \frac{d\hat{b}}{ds} \perp \hat{b}$

We have obtained: $\frac{d\hat{b}}{ds} \perp \hat{t}$ and $\frac{d\hat{b}}{ds} \perp \hat{b}$

$\Rightarrow \frac{d\hat{b}}{ds} \parallel \hat{t}\times\hat{b} \Rightarrow \frac{d\hat{b}}{ds} \parallel (-\hat{n}) \Rightarrow \boxed{\frac{d\hat{b}}{ds} = -\tau\hat{n}}$;

where τ is a scalar, called the torsion of the curve.

(ii) $\hat{n} = \hat{b}\times\hat{t}$, as $\hat{t}, \hat{n}, \hat{b}$ form a right-handed system

$\therefore \frac{d\hat{n}}{ds} = \hat{b}\times\frac{d\hat{t}}{ds} + \frac{d\hat{b}}{ds}\times\hat{t} = \hat{b}\times\kappa\hat{n} + (-\tau)\hat{n}\times\hat{t}$

$$= -\kappa\hat{t} + \tau\hat{b}$$

$\therefore \boxed{\frac{d\hat{n}}{ds} = \tau\hat{b} - \kappa\hat{t}.}$

- Find the expressions for:- Curvature (κ), torsion (τ), Equations of 3 fundamental Planes.

1. Eqn. of the space curve: $\boxed{\bar{r} = \bar{r}(t)}$

$$\Rightarrow \dot{\bar{r}} = \frac{d\bar{r}}{dt} = \frac{d\bar{r}}{ds}\frac{ds}{dt} = \hat{t}\dot{s}, \quad |\dot{\bar{r}}| = |\dot{s}| \longrightarrow (1)$$

$$\Rightarrow \ddot{\bar{r}} = \frac{d\hat{t}}{dt}\dot{s} + \hat{t}\ddot{s} = \frac{d\hat{t}}{ds}\frac{ds}{dt}\dot{s} + \hat{t}\ddot{s} = \kappa\hat{n}\dot{s}^2 + \hat{t}\ddot{s} \quad \left[\because \frac{d\hat{t}}{ds} = \kappa\hat{n}\right]$$

$$\Rightarrow \dddot{\bar{r}} = \frac{d\kappa}{dt}\hat{n}\dot{s}^2 + \kappa\frac{d\hat{n}}{dt}\dot{s}^2 + \kappa\hat{n}\,2\dot{s}\ddot{s} + \frac{d\hat{t}}{dt}\ddot{s} + \hat{t}\dddot{s}$$

$$= \dot{\kappa}\hat{n}\dot{s}^2 + \kappa\frac{d\hat{n}}{ds}\dot{s}^3 + 2\kappa\hat{n}\dot{s}\ddot{s} + \frac{d\hat{t}}{ds}\dot{s}\ddot{s} + \hat{t}\dddot{s}$$

$$= \dot{\kappa}\hat{n}\dot{s}^2 + \kappa\dot{s}^3(\tau\hat{b} - \kappa\hat{t}) + 2\kappa\hat{n}\dot{s}\ddot{s} + \kappa\hat{n}\dot{s}\ddot{s} + \hat{t}\dddot{s}$$

$$= \dot{\kappa}\hat{n}\dot{s}^2 + \kappa\dot{s}^3(\tau\hat{b} - \kappa\hat{t}) + 3\kappa\hat{n}\dot{s}\ddot{s} + \hat{t}\dddot{s}$$

$$\therefore \dot{\bar{r}} \times \ddot{\bar{r}} = \hat{t}\dot{s} \times (\kappa\hat{n}\dot{s}^2 + \hat{t}\ddot{s}) = \kappa\dot{s}^3(\hat{t}\times\hat{n}) + \dot{s}\ddot{s}(\hat{t}\times\hat{t})$$

$$= \kappa\dot{s}^3\hat{b} + \bar{0} = \kappa\dot{s}^3\hat{b} \longrightarrow (2)$$

And
$$\dot{\bar{r}} \times \ddot{\bar{r}} \cdot \dddot{\bar{r}} = \kappa\dot{s}^3\hat{b} \cdot \{\dot{\kappa}\hat{n}\dot{s}^2 + \kappa\dot{s}^3(\tau\hat{b} - \kappa\hat{t}) + 3\kappa\hat{n}\dot{s}\ddot{s} + \hat{t}\dddot{s}\}$$

$$= \kappa\dot{s}^3\hat{b} \cdot \{(\dot{\kappa}\dot{s}^2 + 3\kappa\dot{s}\ddot{s})\hat{n} + \kappa\dot{s}^3\tau\hat{b} + (\dddot{s} - \kappa^2\dot{s}^3)\hat{t}\}$$

$$= \kappa\dot{s}^3\kappa\dot{s}^3\tau(\hat{b}\cdot\hat{b}) = \kappa^2\dot{s}^6\tau \longrightarrow (3)$$

From (2), $|\dot{\bar{r}} \times \ddot{\bar{r}}| = |\kappa\dot{s}^3\hat{b}| = \kappa|\dot{s}|^3 = \kappa|\dot{\bar{r}}|^3$ [by (1)]

$$\Rightarrow \boxed{\kappa = \frac{|\dot{\bar{r}} \times \ddot{\bar{r}}|}{|\dot{\bar{r}}|^3}}$$

From (3), $\dfrac{\dot{\bar{r}} \times \ddot{\bar{r}} \cdot \dddot{\bar{r}}}{\kappa^2|\dot{\bar{r}}|^6} = \tau \Rightarrow \boxed{\tau = \dfrac{\dot{\bar{r}} \times \ddot{\bar{r}} \cdot \dddot{\bar{r}}}{|\dot{\bar{r}} \times \ddot{\bar{r}}|^2}}$

From (1), we see $\hat{t}$ is in direction of $\dot{\bar{r}}$,

From (2), " " $\hat{b}$ " " " " $\dot{\bar{r}} \times \ddot{\bar{r}}$,

And since $\hat{n} = \hat{b} \times \hat{t}$, $\hat{n}$ is in direction of $(\dot{\bar{r}} \times \ddot{\bar{r}}) \times \dot{\bar{r}}$

$\therefore$ Eqn of the Osculating Plane: $(\bar{r} - \bar{r}_0) \cdot \dot{\bar{r}} \times \ddot{\bar{r}} = 0$,

" " " Normal " : $(\bar{r} - \bar{r}_0) \cdot \dot{\bar{r}} = 0$,

" " " Rectifying " : $(\bar{r} - \bar{r}_0) \cdot (\dot{\bar{r}} \times \ddot{\bar{r}}) \times \dot{\bar{r}} = 0$.

2. Eqn of the space curve: $\boxed{\bar{r} = \bar{r}(s)}$.

$$\Rightarrow \bar{r}' = \frac{d\bar{r}}{ds} = \hat{t}$$

$$\Rightarrow \bar{r}'' = \frac{d\hat{t}}{ds} = \kappa\hat{n} \Rightarrow \kappa|\hat{n}| = |\bar{r}''| \Rightarrow \boxed{\kappa = |\bar{r}''|}$$

$$\Rightarrow \bar{r}''' = \frac{d\kappa}{ds}\hat{n} + \kappa\frac{d\hat{n}}{ds} = \kappa'\hat{n} + \kappa(\tau\hat{b} - \kappa\hat{t})$$

Now $\bar{r}' \times \bar{r}'' = \kappa\hat{t}\times\hat{n} = \kappa\hat{b}$

And $(\bar{r}'\times\bar{r}'')\cdot\bar{r}''' = \kappa\hat{b}\cdot\{\kappa'\hat{n} + \kappa(\tau\hat{b} - \kappa\hat{t})\}$

$$= \kappa\kappa' 0 + \kappa^2\tau(\hat{b}\cdot\hat{b}) - \kappa^2(\hat{b}\cdot\hat{t})$$

$$= \kappa^2\tau$$

$$\therefore \tau = \frac{\bar{r}'\times\bar{r}''\cdot\bar{r}'''}{\kappa^2} \Rightarrow \boxed{\tau = \frac{\bar{r}'\times\bar{r}''\cdot\bar{r}'''}{|\bar{r}''|^2}}.$$

Here $\hat{t}$ is in direction of $\bar{r}'$.

$\hat{b}$ " " " " $\bar{r}'\times\bar{r}''$

and $\hat{n}$ " " " " $(\bar{r}'\times\bar{r}'')\times\bar{r}'$

$\therefore$ Eqn of the osculating Plane: $(\bar{r}-\bar{r}_0)\cdot\bar{r}'\times\bar{r}'' = 0$,

" " " Normal " : $(\bar{r}-\bar{r}_0)\cdot\bar{r}' = 0$,

" " " Rectifying " : $(\bar{r}-\bar{r}_0)\cdot(\bar{r}'\times\bar{r}'')\times\bar{r}' = 0$

Ex: Find the curvature (κ) and torsion (τ) for the plane curves: (i) $x = x(t)$, $y = y(t)$, $z = 0$.
(ii) $y = f(x)$, $z = 0$.

(i) $\bar{r} = \bar{r}(t) = x(t)\hat{i} + y(t)\hat{j}$,

$$\Rightarrow \dot{\bar{r}} = \dot{x}\hat{i} + \dot{y}\hat{j} \Rightarrow \ddot{\bar{r}} = \ddot{x}\hat{i} + \ddot{y}\hat{j} \Rightarrow \dddot{\bar{r}} = \dddot{x}\hat{i} + \dddot{y}\hat{j}$$

$$\kappa = \frac{|\dot{\bar{r}}\times\ddot{\bar{r}}|}{|\dot{\bar{r}}|^3} = \frac{|(\dot{x}\ddot{y} - \ddot{x}\dot{y})\hat{k}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} \longrightarrow (1)$$

$$\dot{\bar{r}}\times\ddot{\bar{r}}\cdot\dddot{\bar{r}} = (\dot{x}\ddot{y} - \ddot{x}\dot{y})\hat{k}\cdot(\dddot{x}\hat{i} + \dddot{y}\hat{j}) = 0.$$

$$\therefore \tau = \frac{\dot{\bar{r}}\times\ddot{\bar{r}}\cdot\dddot{\bar{r}}}{|\dot{\bar{r}}\times\ddot{\bar{r}}|^2} = 0. \longrightarrow (2)$$

(ii) Let $x = t$, $y = f(t)$. Then $\kappa = \dfrac{|\ddot{y}|}{(1+\dot{y}^2)^{3/2}}$; $\tau = 0$. $\boxed{\text{From } (1) \& (2)}$

Ex. A particle moves along the curve; $\bar{r} = (t^3-4t)\hat{i} + (t^2+4t)\hat{j} + (8t^2-3t^3)\hat{k}$, where t is the time. Show that its accelation $\bar{a}$ is given by $\bar{a} = \frac{dv}{dt}\hat{t} + \frac{v^2}{\rho}\hat{n}$, where v is the magnitude of ~~its~~ velocity $\bar{v}$, $\rho = 1/K$.
Find also the magnitudes of the tangential and normal components of $\bar{a}$ when $t=2$.

$\bar{r} = \bar{r}(t) \Rightarrow \dot{\bar{r}} = \bar{v} \Rightarrow \bar{v} = \dot{\bar{r}} = \frac{d\bar{r}}{ds}\frac{ds}{dt} = \hat{t}\,v$, $\left[v = \frac{ds}{dt} = |\dot{\bar{r}}|\right]$

$\Rightarrow \frac{d\bar{v}}{dt} = \bar{a} = \frac{d\hat{t}}{dt}v + \hat{t}\frac{dv}{dt} = \hat{t}\frac{dv}{dt} + \frac{d\hat{t}}{ds}\frac{ds}{dt}v$

$\Rightarrow \bar{a} = \frac{dv}{dt}\hat{t} + v^2 K\hat{n} = \frac{dv}{dt}\hat{t} + \frac{v^2}{\rho}\hat{n}$.

$\therefore \boxed{\bar{a} = \frac{dv}{dt}\hat{t} + \frac{v^2}{\rho}\hat{n}}$

So the magnitude of the tangential component of $\bar{a}$ is $\frac{dv}{dt}$; and " " " normal " " $\bar{a}$ is v^2/ρ.

Now $\bar{r} = (t^3-4t,\ t^2+4t,\ 8t^2-3t^3)$

$\Rightarrow \dot{\bar{r}} = \bar{v} = (3t^2-4,\ 2t+4,\ 16t-9t^2)$

$\Rightarrow \ddot{\bar{r}} = \frac{d\bar{v}}{dt} = (6t, 2, 16-18t)$

$v = |\bar{v}| = \sqrt{(3t^2-4)^2+(2t+4)^2+(16t-9t^2)^2} = \sqrt{144} = 12$ at $t=2$.

$K = \frac{|\dot{\bar{r}} \times \ddot{\bar{r}}|}{|\dot{\bar{r}}|^3}$, where

$\dot{\bar{r}} = (3.2^2-4,\ 2.2+4,\ 16.2-9.2^2) = (8, 8, -4)$ at $t=2$.

$\ddot{\bar{r}} = (6.2, 2, 16-18.2) = (12, 2, -20)$ at $t=2$.

Now $\dot{\bar{r}} \times \ddot{\bar{r}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 8 & 8 & -4 \\ 12 & 2 & -20 \end{vmatrix} = (-152, 112, -80) = 8(-19, 14, -10)$

$\therefore K = \frac{|8(-19,14,-10)|}{|(8,8,-4)|^3} = \frac{8\sqrt{(19)^2+(14)^2+(10)^2}}{4^3(\sqrt{2^2+2^2+1})^3} = \frac{8\sqrt{361+196+100}}{(12)^3} = \frac{8\sqrt{657}}{(12)^3}$.

Now $\frac{dv}{dt} = \frac{1}{2v} \times [2(3t^2-4)\times 6t + 2(2t+4).2 + 2(16t-9t^2).(16-18t)]$

$= \frac{1}{2\times 12} \times (192+32+160) = 16$.

$\therefore \quad \frac{v^2}{\rho} = v^2\kappa = (12)^2 \cdot \frac{8\sqrt{657}}{(12)^3} = \frac{2}{3}\sqrt{657} = \frac{2}{3}\times 3\sqrt{73}$

$= 2\sqrt{73}$.

$\therefore$ Tangential component of $\bar{a} = \frac{dv}{dt} = 16$,

& Normal " " $\bar{a} = v^2/\rho = 2\sqrt{73}$,

when $t=2$.

Ex. 53 (Spiegel). Show that the Frenet-Serret formulae can be written in the form:
$\frac{d\hat{t}}{ds} = \bar{\omega}\times\hat{t}$, $\frac{d\hat{n}}{ds} = \bar{\omega}\times\hat{n}$, $\frac{d\hat{b}}{ds} = \bar{\omega}\times\hat{b}$; Determine ω.

Frenet-Serret formulae:

$\frac{d\hat{t}}{ds} = \kappa\hat{n} \longrightarrow (1)$; $\frac{d\hat{n}}{ds} = \tau\hat{b} - \kappa\hat{t} \longrightarrow (2)$; $\frac{d\hat{b}}{ds} = -\tau\hat{n} \rightarrow (3)$.

From (2), $\frac{d\hat{n}}{ds} = \tau\hat{b} - \kappa\hat{t} = \tau(\hat{t}\times\hat{n}) + \kappa(\hat{b}\times\hat{n}) = (\tau\hat{t} + \kappa\hat{b})\times\hat{n}$

$= \bar{\omega}\times\hat{n}$, where $\bar{\omega} = \tau\hat{t} + \kappa\hat{b}$.

$\therefore \frac{d\hat{n}}{ds} = \bar{\omega}\times\hat{n}$

From (1), $\frac{d\hat{t}}{ds} = \kappa\hat{n} = \kappa(\hat{b}\times\hat{t}) = (\kappa\hat{b} + \tau\hat{t})\times\hat{t} = \bar{\omega}\times\hat{t}$.

From (3), $\frac{d\hat{b}}{ds} = -\tau\hat{n} = -\tau(\hat{b}\times\hat{t}) = \tau(\hat{t}\times\hat{b})$

$= (\tau\hat{t} + \kappa\hat{b})\times\hat{b} = \bar{\omega}\times\hat{b}$.

$\therefore \frac{d\hat{b}}{ds} = \bar{\omega}\times\hat{b}$

And $\bar{\omega} = \tau\hat{t} + \kappa\hat{b}$.

Ex. 69. Prove that the acceleration vector of a particle moving along a space curve always lies in the osculating plane.

We have: $\bar{a} = \frac{dv}{dt}\hat{t} + \frac{v^2}{\rho}\hat{n}$

or, $\bar{a}\cdot\hat{t}\times\hat{n} = \frac{dv}{dt}(\hat{t}\cdot\hat{t}\times\hat{n}) + \frac{v^2}{\rho}(\hat{n}\cdot\hat{t}\times\hat{n}) = 0 + 0 = 0$

$\therefore$ $\bar{a}, \hat{t}, \hat{n}$ are coplanar $\Rightarrow \bar{a}$ lies in the plane containing $\hat{t}$ and $\hat{n}$.

$\Rightarrow \bar{a}$ lies in the osculating plane.

Ex. 64. Find the equations for the tangent plane and normal line to the surface $4z = x^2 - y^2$ at $(3,1,2)$.

Let $x = u,\ y = v,\ z = \frac{u^2 - v^2}{4}$ be the parametric equations representing the given surface.

$\therefore\ \ \bar{r} = \bar{r}(u,v) = u\hat{i} + v\hat{j} + \frac{u^2-v^2}{4}\hat{k}$

Normal vector to the surface is given by

$$\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} = \left(\hat{i} + \frac{u}{2}\hat{k}\right) \times \left(\hat{j} - \frac{v}{2}\hat{k}\right) = -\frac{u}{2}(\hat{j}\times\hat{k}) + \frac{v}{2}(\hat{k}\times\hat{i}) + (\hat{i}\times\hat{j}) - \frac{uv}{4}(\hat{k}\times\hat{k}).$$

$$= -\frac{u}{2}\hat{i} + \frac{v}{2}\hat{j} + \hat{k}$$

$(x,y,z) = (3,1,2) \Rightarrow u = 3,\ v = 1$

$\therefore$ At $(3,1,2)$, $\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} = -\frac{3}{2}\hat{i} + \frac{1}{2}\hat{j} + \hat{k} = \bar{n}$, say.

Equation of the tangent plane at $(3,1,2)$ is given by

$$\{\bar{r} - (3,1,2)\}\cdot\bar{n} = 0 \Rightarrow \{(x,y,z) - (3,1,2)\}\cdot\left(-\tfrac{3}{2}, \tfrac{1}{2}, 1\right) = 0$$

$$\Rightarrow -\frac{3}{2}(x-3) + \frac{1}{2}(y-1) + 1\cdot(z-2) = 0 \Rightarrow -3(x-3) + y - 1 + 2(z-2) = 0$$

$$\Rightarrow -3x + 9 + y - 1 + 2z - 4 = 0 \Rightarrow \underline{3x - y - 2z = 4}.$$

Equation of the normal line at $(3,1,2)$ is given by

$$\{\bar{r} - (3,1,2)\}\times\bar{n} = \bar{0} \Rightarrow \{(x,y,z) - (3,1,2)\}\times\left(-\tfrac{3}{2}, \tfrac{1}{2}, 1\right) = \bar{0}$$

$$\Rightarrow \frac{x-3}{-3/2} = \frac{y-1}{1/2} = \frac{z-2}{1} \Rightarrow \underline{\frac{x-3}{-3} = \frac{y-1}{1} = \frac{z-2}{2}}$$

In parametric form: $\frac{x-3}{-3} = \frac{y-1}{1} = \frac{z-2}{2} = t$ (say).

$$\Rightarrow \underline{x = 3 - 3t,\ \ y = 1 + t,\ \ z = 2 + 2t}.$$

Ex. Prove that the radius of curvature ρ on a space curve is given by $\rho = \frac{v^3}{|\bar{v}\times\bar{a}|}$.

Let $\bar{r} = \bar{r}(t)$ be the eqn. of the space curve.

Then $\dot{\bar{r}} = \bar{v}$, $|\dot{\bar{r}}| = |\bar{v}| = v$; $\ddot{\bar{r}} = \frac{d\bar{v}}{dt} = \bar{a}$

$\bar{v} = \frac{d\bar{r}}{ds}\frac{ds}{dt} = \hat{t}\,|\dot{\bar{r}}| = v\hat{t} \Rightarrow \frac{d\bar{v}}{dt} = \frac{d\hat{t}}{dt}v + \hat{t}\frac{dv}{dt}$

$\Rightarrow \bar{a} = \frac{d\hat{t}}{ds}\frac{ds}{dt}v + \hat{t}\frac{dv}{dt} = \kappa\hat{n}v^2 + \hat{t}\frac{dv}{dt}$

$\Rightarrow \bar{v}\times\bar{a} = v\hat{t}\times\left(\kappa\hat{n}v^2 + \hat{t}\frac{dv}{dt}\right) = \kappa v^3\hat{b} \Rightarrow |\bar{v}\times\bar{a}| = \kappa v^3 \Rightarrow \rho = \frac{v^3}{|\bar{v}\times\bar{a}|}$

Ex. Given the space curve $x=t,\ y=t^2,\ z=\frac{2}{3}t^3$, find the curvature κ and the torsion τ.

$$\bar{r}=\bar{r}(t)=t\,\hat{i}+t^2\hat{j}+\tfrac{2}{3}t^3\hat{k}$$

$$\dot{\bar{r}}=\hat{i}+2t\hat{j}+2t^2\hat{k}\ ;\quad \ddot{\bar{r}}=2\hat{j}+4t\hat{k}$$

$$\dot{\bar{r}}\times\ddot{\bar{r}}=(\hat{i}+2t\hat{j}+2t^2\hat{k})\times(2\hat{j}+4t\hat{k})$$

$$=2(\hat{i}\times\hat{j})+2t(\hat{j}\times\hat{j})+4t^2(\hat{k}\times\hat{j})+4t(\hat{i}\times\hat{k})$$

$$+8t^2(\hat{j}\times\hat{k})+8t^3(\hat{k}\times\hat{k})$$

$$=2\hat{k}-4t^2\hat{i}-4t\hat{j}+8t^2\hat{i}=4t^2\hat{i}-4t\hat{j}+2\hat{k}$$

$$\therefore\ |\dot{\bar{r}}|=\sqrt{1^2+4t^2+4t^4}=\sqrt{(2t^2+1)^2}=2t^2+1$$

And $$|\dot{\bar{r}}\times\ddot{\bar{r}}|=\sqrt{(4t^2)^2+(4t)^2+2^2}=\sqrt{16t^4+16t^2+4}=4t^2+2$$

$$=2(2t^2+1).$$

$$\therefore\ \kappa=\frac{|\dot{\bar{r}}\times\ddot{\bar{r}}|}{|\dot{\bar{r}}|^3}=\frac{2(2t^2+1)}{(2t^2+1)^3}=\frac{2}{(2t^2+1)^2}.$$

Now $$\tau=\frac{\dot{\bar{r}}\times\ddot{\bar{r}}\cdot\dddot{\bar{r}}}{|\dot{\bar{r}}\times\ddot{\bar{r}}|^2}\ ;\quad \dddot{\bar{r}}=4\hat{k}\ ;\quad \dot{\bar{r}}\times\ddot{\bar{r}}\cdot\dddot{\bar{r}}=8$$

$$=\frac{8}{2^2(2t^2+1)^2}=\frac{2}{(2t^2+1)^2}=\kappa$$

$$\therefore\ \kappa=\tau=\frac{2}{(2t^2+1)^2}.$$

Also find the equations of the osculating plane, normal plane and the rectifying plane for the above curve at $t=1$.

At $t=1$, $\dot{\bar{r}}=2\hat{j}+2\hat{k}$, $\ddot{\bar{r}}=2\hat{j}+4\hat{k}$, $\dot{\bar{r}}\times\ddot{\bar{r}}=4\hat{i}-4\hat{j}+2\hat{k}$.

Now $$(\dot{\bar{r}}\times\ddot{\bar{r}})\times\dot{\bar{r}}=(4\hat{i}-4\hat{j}+2\hat{k})\times(2\hat{j}+2\hat{k})$$

$$=8(\hat{i}\times\hat{j})-8(\hat{j}\times\hat{j})+4(\hat{k}\times\hat{j})+8(\hat{i}\times\hat{k})$$

$$-8(\hat{j}\times\hat{k})+4(\hat{k}\times\hat{k})=-12\hat{i}-8\hat{j}+8\hat{k}$$

Eqn of osculating plane is $(\bar{r}-\bar{r}_0)\cdot(\dot{\bar{r}}\times\ddot{\bar{r}})=0$

$$\Rightarrow \{(x,y,z)-(1,1,2/3)\}\cdot(4,-4,2)=0$$

$$\Rightarrow 4(x-1)-4(y-1)+2(z-2/3)=0$$

$$\Rightarrow 4x-4y+2z=4/3.$$

Equation of the Normal plane is $(\bar{r}-\bar{r}_0)\cdot\dot{\bar{r}}=0$

$\Rightarrow \{(x-1),(y-1),(z-2/3)\}\cdot(0,2,2)=0$

$\Rightarrow 2(y-1)+2(z-2/3)=0 \Rightarrow 2y+2z=10/3$

$\Rightarrow \underline{y+z=5/3}$.

Equation of the Rectifying plane is

$(\bar{r}-\bar{r}_0)\cdot(\dot{\bar{r}}\times\ddot{\bar{r}})\times\dot{\bar{r}}=0$

$\Rightarrow (x-1, y-1, z-2/3)\cdot(-12,-8,8)=0$

$\Rightarrow -3(x-1)-2(y-1)+2(z-2/3)=0$

$\Rightarrow 3x+2y-2z=11/3$.

Also find the equations of the Tangent line, Principal normal and Binormal for the above curve at $t=1$.

Unit tangent $\hat{t}$ is in the direction of $\dot{\bar{r}}$.

Principal normal $\hat{n}$ is in the direction of $(\dot{\bar{r}}\times\ddot{\bar{r}})\times\dot{\bar{r}}$

And Binormal $\hat{b}$ " " " " " $\dot{\bar{r}}\times\ddot{\bar{r}}$

$\therefore$ Equation of the tangent at $t=1$ is given by

$(\bar{r}-\bar{r}_0)\times\dot{\bar{r}}=\bar{0} \Rightarrow (x-1, y-1, z-2/3)\times(0,2,2)=\bar{0}$

$\Rightarrow \dfrac{x-1}{0}=\dfrac{y-1}{2}=\dfrac{z-2/3}{2}=t$ (say)

$\Rightarrow x=1,\ y=1+2t,\ z=2/3+2t$.

Equation of the Principal normal is given by

$(\bar{r}-\bar{r}_0)\times\{(\dot{\bar{r}}\times\ddot{\bar{r}})\times\dot{\bar{r}}\}=\bar{0} \Rightarrow (x-1, y-1, z-2/3)\times(-12,-8,8)=\bar{0}$

$\Rightarrow \dfrac{x-1}{-12}=\dfrac{y-1}{-8}=\dfrac{z-2/3}{8}=t$ (say) $\Rightarrow x=1-12t,\ y=1-8t,\ z=2/3+8t$.

Equation of the Binormal is given by

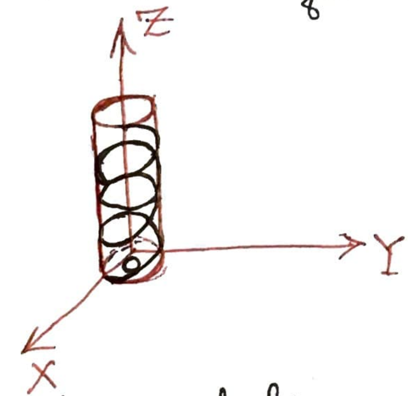
$(\bar{r}-\bar{r}_0)\times(\dot{\bar{r}}\times\ddot{\bar{r}})=\bar{0}$

$\Rightarrow (x-1, y-1, z-2/3)\times(4,-4,2)=\bar{0}$

$\Rightarrow \dfrac{x-1}{4}=\dfrac{y-1}{-4}=\dfrac{z-2/3}{2} \Rightarrow x=1+4t,\ y=1-4t,\ z=2/3+2t$.

(parametric equations).

Circular Helix :- The parametric equations $x = a\cos t$, $y = a\sin t$, $z = bt$ $(b \neq 0)$ represent a curve on the surface of a circular cylinder and cutting the generators at a constant angle. This curve is known as a circular helix. This curve is twisted, i.e., does not lie in a plane. And the curve is described by a moving point P which is subjected to a constant rotation together with a translation in the direction of the axis of rotation



Verify for the circular helix, $\kappa = \frac{a}{a^2+b^2}$; $\tau = \frac{b}{a^2+b^2}$.

Eqns of circular helix: $x = a\cos t$, $y = a\sin t$, $z = bt$.

$\bar{r}(t) = (a\cos t,\ a\sin t,\ bt)$, $\dot{\bar{r}} = (-a\sin t,\ a\cos t,\ b)$

$\ddot{\bar{r}} = (-a\cos t,\ -a\sin t,\ 0)$, $\dddot{\bar{r}} = (a\sin t,\ -a\cos t,\ 0)$

Now $\dot{\bar{r}} \times \ddot{\bar{r}} = (-a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\hat{k}) \times (-a\cos t\,\hat{i} - a\sin t\,\hat{j})$

$= -ab\cos t\,\hat{j} + (a^2\sin^2 t + a^2\cos^2 t)\,\hat{k} + ab\sin t\,\hat{i}$

$= ab\sin t\,\hat{i} - ab\cos t\,\hat{j} + a^2\hat{k}$.

And $\dot{\bar{r}} \times \ddot{\bar{r}} \cdot \dddot{\bar{r}} = a^2b\sin^2 t + a^2b\cos^2 t = a^2b$

$|\dot{\bar{r}} \times \ddot{\bar{r}}| = (a^2b^2 + a^4)^{1/2} = a(a^2+b^2)^{1/2}$.

$$\kappa = \frac{|\dot{\bar{r}} \times \ddot{\bar{r}}|}{|\dot{\bar{r}}|^3} = \frac{a(a^2+b^2)^{1/2}}{(a^2+b^2)^{3/2}} = \frac{a}{a^2+b^2}.$$

$$\tau = \frac{\dot{\bar{r}} \times \ddot{\bar{r}} \cdot \dddot{\bar{r}}}{|\dot{\bar{r}} \times \ddot{\bar{r}}|^2} = \frac{a^2b}{a^2(a^2+b^2)} = \frac{b}{a^2+b^2}.$$

Also find the length of the curve measured from $t=0$ to some point t is $t\sqrt{a^2+b^2}$.

We have $\dot{\bar{r}} = |\dot{\bar{r}}|\,\hat{t}$, where $|\dot{\bar{r}}| = \frac{ds}{dt}$, $\hat{t} = \frac{d\bar{r}}{ds}$

Now $|\dot{\bar{r}}| = \frac{ds}{dt} = \sqrt{a^2+b^2} \Rightarrow \int_0^s ds = \int_0^t \sqrt{a^2+b^2}\,dt$

$\Rightarrow s = t\sqrt{a^2+b^2}$.

Ex: Prove that the necessary and sufficient condition that a curve $\bar{r} = \bar{f}(s)$ be a helix is that $\bar{r}'' \cdot \bar{r}''' \times \bar{r}'''' = 0$.

We know that—

a <u>helix</u> is a twisted curve whose tangent makes a constant angle with a fixed line called its axis.

So that if the unit vector $\hat{e}$ has the direction of its axis, then the equ of the helix is given by

$$\hat{e} \cdot \hat{t} = \cos\alpha \quad (0 < \alpha < \pi/2), \ \alpha \text{ is the constant.}$$

We shall first prove that—

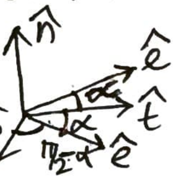
For a helix, $\frac{K}{\tau} = \pm \tan\alpha$ (necessary & sufficient) (i.e. K/τ is constant)

$\hat{e} \cdot \hat{t} = \cos\alpha$

Differentiating w.r.t. s, we get

$\hat{e} \cdot \frac{d\hat{t}}{ds} = 0$ [since $\hat{e}$ is a constant vector & α is also constant]

$\Rightarrow \hat{e} \cdot K\hat{n} = 0 \Rightarrow \hat{e} \cdot \hat{n} = 0$. $\boxed{\hat{b} \cdot \hat{e} = \cos(\frac{\pi}{2} \pm \alpha) = \pm \sin\alpha}$



$\Rightarrow \hat{n} \perp \hat{e}$. $\Rightarrow \hat{b}, \hat{t}, \hat{e}$ are coplanar,

So that $\hat{b} \cdot \hat{e} = \pm \sin\alpha \Rightarrow \frac{d\hat{b}}{ds} \cdot \hat{e} = 0 \Rightarrow -\tau\hat{n} \cdot \hat{e} = 0$

$\Rightarrow \frac{d\hat{n}}{ds} \cdot \hat{e} = 0 \Rightarrow (\tau\hat{b} - K\hat{t}) \cdot \hat{e} = 0$

$\Rightarrow \tau(\hat{b} \cdot \hat{e}) - K(\hat{t} \cdot \hat{e}) = 0 \Rightarrow \tau(\pm \sin\alpha) - K(\cos\alpha) = 0$

$\Rightarrow K/\tau = \pm \tan\alpha \Rightarrow K/\tau$ is constant.

Conversely, let K/τ be constant $= c$ (say)

So $\frac{d\hat{t}}{ds} = K\hat{n}$ and $\frac{d\hat{b}}{ds} = -\tau\hat{n} \Rightarrow \frac{d\hat{t}}{ds} = \left(\frac{K}{\tau}\right)\frac{d\hat{b}}{ds}$

$\Rightarrow \frac{d\hat{t}}{ds} = -c\frac{d\hat{b}}{ds} \Rightarrow \hat{t} = -c\hat{b} + \bar{c}_1$ [by integrating]

$\Rightarrow \hat{t} \cdot \hat{t} = -c\hat{b} \cdot \hat{t} + \bar{c}_1 \cdot \hat{t} \Rightarrow 1 = 0 + \bar{c}_1 \cdot \hat{t}$ [$\bar{c}_1$ is constant]

$\Rightarrow |\bar{c}_1| \cos\theta = 1 \Rightarrow \theta = \text{Constant}$

$\Rightarrow$ Tangent makes a constant angle with the direction of the fixed vector $\bar{c}_1$.

$\therefore$ Necessary and sufficient condition for a curve to be a helix is that $K/\tau = (\pm)\tan\alpha$ [constant]

Necessary Part $\longrightarrow$

Let $\bar{r} = \bar{f}(s)$ be a helix. Then $\kappa/\tau = \pm \tan\alpha$.

Now $\bar{r}' = \frac{d\bar{r}}{ds} = \hat{t}$, $\bar{r}'' = \frac{d\hat{t}}{ds} = \kappa\hat{n}$, $\bar{r}''' = \frac{d\kappa}{ds}\hat{n} + \kappa\frac{d\hat{n}}{ds}$

$\therefore\ \bar{r}''' = \frac{d\kappa}{ds}\hat{n} + \kappa(\tau\hat{b} - \kappa\hat{t})$

$\Rightarrow \bar{r}'''' = \frac{d^2\kappa}{ds^2}\hat{n} + \frac{d\kappa}{ds}\frac{d\hat{n}}{ds} + \frac{d\kappa}{ds}(\tau\hat{b} - \kappa\hat{t}) + \kappa\left[\frac{d\tau}{ds}\hat{b} + \tau\frac{d\hat{b}}{ds} - \frac{d\kappa}{ds}\hat{t} - \kappa\frac{d\hat{t}}{ds}\right] = \kappa''\hat{n} + \kappa'(\tau\hat{b} - \kappa\hat{t}) + \kappa'(\tau\hat{b} - \kappa\hat{t}) + \kappa[\tau'\hat{b} + \tau(-\tau\hat{n}) - \kappa'\hat{t} - \kappa(\kappa\hat{n})]$

$= (\kappa'' - \tau^2\kappa - \kappa^3)\hat{n} + (2\kappa'\tau + \kappa\tau')\hat{b} - 3\kappa'\kappa\hat{t}$

$\bar{r}'' \times \bar{r}''' = \kappa^2\tau(\hat{n}\times\hat{b}) - \kappa^3\hat{n}\times\hat{t} = \kappa^2\tau\hat{t} + \kappa^3\hat{b}$

So $\bar{r}''\cdot\bar{r}'''\times\bar{r}'''' = \bar{r}''\times\bar{r}'''\cdot\bar{r}'''' = -3\kappa^3\kappa'\tau + \kappa^3(2\kappa'\tau + \kappa\tau')$

$= -\kappa^3\kappa'\tau + \kappa^4\tau' = \kappa^3(\kappa\tau' - \kappa'\tau)$

$= \kappa^3\cdot\kappa^2\left(\frac{\kappa\tau' - \kappa'\tau}{\kappa^2}\right) = \kappa^5\frac{d}{ds}\left(\frac{\tau}{\kappa}\right)$

For a helix, $\frac{\tau}{\kappa}$ is constant $\Rightarrow \frac{d}{ds}\left(\frac{\tau}{\kappa}\right) = 0$

$\therefore\ \underline{\bar{r}''\cdot\bar{r}'''\times\bar{r}'''' = 0}$ if $\bar{r} = \bar{f}(s)$ be a helix.

Sufficient Part $\longrightarrow$

Let $\bar{r}''\cdot\bar{r}'''\times\bar{r}'''' = 0$ for a curve $\bar{r} = \bar{f}(s)$.

Since $\bar{r}''\cdot\bar{r}'''\times\bar{r}'''' = \kappa^5\frac{d}{ds}\left(\frac{\tau}{\kappa}\right) = 0$ [given]

$\Rightarrow \frac{\tau}{\kappa} = \text{Constant}$

$\Rightarrow \underline{\bar{r} = \bar{f}(s) \text{ is a helix.}}$

Ex. If the space curve is given by $x = 3\cos t$, $y = 3\sin t$, $z = 4t$; find $\hat{t}$, $\hat{n}$, $\hat{b}$, κ, ρ, τ and σ.

$\bar{r} = \bar{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$; it is a helix (circular).

$= 3\cos t\,\hat{i} + 3\sin t\,\hat{j} + 4t\,\hat{k}$.

$\dot{\bar{r}} = (-3\sin t, 3\cos t, 4)$, $|\dot{\bar{r}}| = \sqrt{9(\sin^2 t + \cos^2 t) + 16} = 5$

$\ddot{\bar{r}} = (-3\cos t, -3\sin t, 0)$, $\dddot{\bar{r}} = (3\sin t, -3\cos t, 0)$

$$\ddot{\bar{r}} \times \dddot{\bar{r}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3\cos t & -3\sin t & 0 \\ 3\sin t & -3\cos t & 0 \end{vmatrix} = 9(\cos^2 t + \sin^2 t)\hat{k} = 9\hat{k}.$$

$\therefore$ $\dot{\bar{r}} \cdot \ddot{\bar{r}} \times \dddot{\bar{r}} = 36$;

$$\dot{\bar{r}} \times \ddot{\bar{r}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3\sin t & 3\cos t & 4 \\ -3\cos t & -3\sin t & 0 \end{vmatrix} = (12\sin t, -12\cos t, 9)$$

$\therefore$ $\kappa = \dfrac{|\dot{\bar{r}} \times \ddot{\bar{r}}|}{|\dot{\bar{r}}|^3} = \dfrac{\sqrt{144+81}}{5^3} = \dfrac{15}{5^3}$

$\therefore$ $\kappa = \dfrac{3}{25}$; $\tau = \dfrac{\dot{\bar{r}} \cdot \ddot{\bar{r}} \times \dddot{\bar{r}}}{|\dot{\bar{r}} \times \ddot{\bar{r}}|^2} = \dfrac{36}{225} = \dfrac{4}{25}$.

Now $\hat{t} = \dfrac{\dot{\bar{r}}}{|\dot{\bar{r}}|} = \dfrac{1}{5}(-3\sin t, 3\cos t, 4) = -\dfrac{3}{5}\sin t\,\hat{i} + \dfrac{3}{5}\cos t\,\hat{j} + 4\hat{k}$.

Since $\dfrac{d\hat{t}}{ds} = \kappa\hat{n} = \dfrac{3}{25}\hat{n} \Rightarrow \hat{n} = \dfrac{25}{3}\dfrac{d\hat{t}}{ds}$.

Now $\dfrac{d\hat{t}}{dt} = \left(-\dfrac{3}{5}\cos t, -\dfrac{3}{5}\sin t, 0\right)$; $\dfrac{d\hat{t}}{ds} = \dfrac{d\hat{t}}{dt} \Big/ \dfrac{ds}{dt} = \dfrac{d\hat{t}}{dt} \Big/ |\dot{\bar{r}}|$

$\Rightarrow$ $\dfrac{d\hat{t}}{ds} = \dfrac{1}{5}\left(-\dfrac{3}{5}\cos t, -\dfrac{3}{5}\sin t, 0\right) = \left(-\dfrac{3}{25}\cos t, -\dfrac{3}{25}\sin t, 0\right)$

$\therefore$ $\hat{n} = \dfrac{25}{3}\dfrac{d\hat{t}}{ds} = (-\cos t, -\sin t, 0)$

$$\hat{b} = \hat{t} \times \hat{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{3}{5}\sin t & \frac{3}{5}\cos t & 4 \\ -\cos t & -\sin t & 0 \end{vmatrix} = (4\sin t, -4\cos t, 3/5)$$

$\therefore$ $\hat{t} = \left(-\dfrac{3}{5}\sin t, \dfrac{3}{5}\cos t, 4\right)$

$\hat{n} = (-\cos t, -\sin t, 0)$

$\hat{b} = (4\sin t, -4\cos t, 3/5)$

$\kappa = \dfrac{3}{25}$, $\rho = \dfrac{1}{\kappa} = \dfrac{25}{3}$; $\tau = \dfrac{4}{25}$.